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I notice that the determination of factors of $f\{m\}$ for $m=9$, 11, 12, 18 and 38, has been recently effected by several English mathematicians. For information regarding these interesting results, see the *Educational Times*, June, 1903, page 270, where reference is made to the announcement by Col. J. A. Cunningham. A summary of the previous work done along this line in given in Ball's *Recreations and Problems*, third edition, pages 34-35.

112. Proposed by L. C. WALKER, A. M., Graduate Student Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5, the first triangle, find the sides of the nth triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and the PROPOSER.

Let $2x_n-1$, $2x_n$, and $2x_n+1$ denote the sides of the nth triangle, r_n the radius of the inscribed circle, and A the area of the triangle.

Then $A=\sqrt{\{[3x_n(x_n+1)(x_n)(x_n-1)]\}}=3x_nr_n$,

$$\text{whence } [x_n-r_n\sqrt{3}][x_n+r_n\sqrt{3}]=1\ldots\ldots[1].$$

Now for the first triangle, $x=2$, and $r=1$; $\therefore [2-\sqrt{3})^n[2+\sqrt{3}]^n=1\ldots\ldots[2]$.

Assume $x_n-r_n\sqrt{3}=[2-\sqrt{3}]^n$; then from [1], $x_n+r_n\sqrt{3}=[2+\sqrt{3}]^n$; from which $x_n=r_n\sqrt{3}=\frac{1}{2}\{[2+\sqrt{3}]^n+[2-\sqrt{3}]^n\}$. Hence,

$$2x_n-1=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n-1,$$

$$2x_n=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n,$$

$$2x_n+1=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n+1,$$

$$A=[\tfrac{1}{4}\sqrt{3}]\{[2+\sqrt{3}]^{2n}-[2-\sqrt{3}]^{2n}\}.$$

II. Solution bp A. H. BELL, Litchfield, Ill.

Take the sides of the rational triangle $x+d$, x, $x-d$; the area of this triangle is $\frac{1}{4}x[3(x^2-4d^2)]^{\frac{1}{2}}$. To have it rational we must have $3[x^2-4d^2]=\square=y^2$, say, in which the area is $\frac{1}{4}xy$. If we make $x=2x'$, and $y=6y'$, the equation becomes, when $d=1$, $x'^2-3y'^2=1\ldots\ldots(2)$. The values of x, by Lagrange, are the numerators of the odd convergents from the $\sqrt{3}$; the first two odd fractions are $\frac{1}{0}$, $\frac{2}{1}$, etc., in which $2x'=x=4$, giving the initial rational triangle with sides 3, 4, 5.

The extension of the values of x', or x, by the algebraic solution of this equation in general, is known to be $x_{n+1}=2x.x_n-x_{n-1}\ldots\ldots(4)$.

This constant factor $2x$, is called by Roberts, and Robins, the magic M, or Key. In this case $M=4$, and the middle side of the next rational triangle is by (4), $x_2=4.4-2=14$, and the triangle 13, 14, 15; by extending the middle side in the same way we get,

No. of triangle:	1	2	3	4	5	6	
	3	13	51	193	723	2701	
Sides:	4	14	52	194	724	2702	etc……(5).
	5	15	53	195	725	2703	

The nth rational triangle can be found by entending formula (4) in terms of M, and x', as follows:

No.	0	1	2	3	4	5	etc.
Series	$x'=1$;	x';	$Mx'-1$;	$[M^2-1]x'-M$;	$[M^3-2m]x'-M^2+1$;	$[M^4-3M^2+1]x'-M^3+2M$;	etc......(6).

By restoring the value of x', above$=2$, we have the series of values of the nth, middle side of the nth rational triangle in terms of the powers of magic M. The quantity in the powers of M multiplied by the constant factor x', I call the "Continued Fraction Theorem" the laws of which are easily determined. These quantities are shown above in brackets in (6), and abbreviated to CFT.

Hence the middle side of the nth triangle$=\{[n\text{th.CFT}]2-(n-1]\text{th.CFT}\}2$.....(7).

The laws of this theorem are: The 1st coefficient of $M=1$.

The 2nd coefficient of $M=$no. cycle -2.

The pth coefficient of $M=$(where $n-p=m$)

$$p\text{th coefficient}=\frac{m[m-1][m-2].....[m-p+2]}{1.2.3.....[p-1]}.....(8).$$

If $n=10$, the 10th.CFT$=[M^9-8M^7+21M^5-20M^3+5M]$, and multiplied by $\frac{1}{2}M=x'=2$, from which take 9th.CFT$=M^8-7M^6+15M^4-10M^2+1$, the result multiplied by 2 will give the middle side, x, of the 10th rational triangle$=524174$.

Then the 10th triangle has for sides 524173, 524174, 524175.

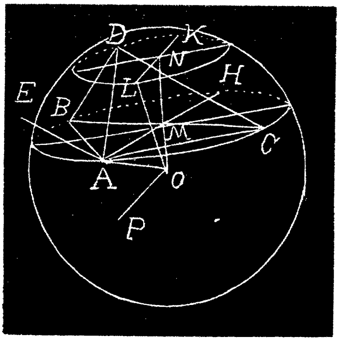
AVERAGE AND PROBABILITY.

130. Proposed by **L. C. WALKER, A. M.,** Graduate Student, Leland Stanford Jr. University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three by three, is 1-35 of the voume of the given sphere.

II. Solution by **G. B. M. ZERR, A. M., Ph.D.,** Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B, C, D be the four random points; AH the diameter of the section through A, B, C; LK the diameter of a section through D parallel to the section through A, B, C; O the center of the sphere, M the center of AH, N the center of LK. Draw AE perpendicular to AH, and OP a line such that AB is parallel to the plane MOP. Let $AO=r$, $\angle AOM=\theta$, $\angle EAC=\varphi$, $\angle EAB=\psi$, $\angle LON=\beta$, $\angle MOP=\lambda$, the angle POM makes with some fixed plane$=\rho$.



Then $AC=2r\sin\theta\sin\varphi$, $AB=2r\sin\theta\sin\psi$; the volume, V, of the tetrahedron $DABC=\frac{2}{3}r^3(\cos\beta-\cos\theta)\sin^2\theta\sin\varphi\sin\psi\sin(\varphi-\psi)$. An element of surface at A is $2\pi r^2\sin\theta d\theta$; at D, $2\pi r^2\sin\beta d\beta$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\lambda\sin\psi d\psi d\rho$. The limits of θ are 0 and π; of β, 0 and θ; of φ, 0 and π; of ψ, 0 and φ and doubled; of λ, 0 and π; of ρ, 0 and 2π. Since the whole number of ways the four points can be taken is $256\pi^4r^8$, we have